

LIEB–THIRING INEQUALITIES ON SOME MANIFOLDS

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Abstract. We prove Lieb–Thirring inequalities with improved constants on the two-dimensional sphere \mathbb{S}^2 and the two-dimensional torus \mathbb{T}^2 . In the one-dimensional periodic case we obtain a simultaneous bound for the negative trace and the number of negative eigenvalues.

Key words: Lieb–Thirring inequalities, Schrödinger operators.
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1. INTRODUCTION

The Schrödinger operator in $L_2(\mathbb{R}^n)$

$$-\Delta + V$$

with a real-valued potential V that sufficiently fast decays at infinity has a discrete negative spectrum satisfying the Lieb–Thirring spectral inequalities [20]

$$\sum_{\nu_i \leq 0} |\nu_i|^\gamma \leq L_{\gamma,n} \int V_-(x)^{\gamma+n/2} dx, \quad (1.1)$$

where $V_\pm(x) = (|V(x)| \pm V(x))/2$. The Lieb–Thirring constant $L_{\gamma,n}$ is finite for $\gamma \geq 1/2$, $n = 1$ (for $\gamma = 1/2$ see [24]); $\gamma > 0$, $n = 2$; and $\gamma \geq 0$, $n \geq 3$ (where $\gamma = 0$ is the Lieb–Cwikel–Rozenblum inequality).

The Lieb–Thirring constants satisfy the lower bound

$$L_{\gamma,n} \geq L_{\gamma,n}^{\text{cl}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 - |\xi|)_+^\gamma dx = \frac{\Gamma(\gamma + 1)}{(4\pi)^{n/2} \Gamma(n/2 + \gamma + 1)}. \quad (1.2)$$

Sharp results valid for all dimensions n , $L_{\gamma,n} = L_{\gamma,n}^{\text{cl}}$, $\gamma \geq 3/2$ were obtained in [18] (see also [5]). The best known estimate of $L_{\gamma,n}$ for $1 \leq \gamma < 3/2$ from [8] is as follows

$$L_{\gamma,n} \leq R \cdot L_{\gamma,n}^{\text{cl}}, \quad R = \frac{\pi}{\sqrt{3}} = 1.8138 \dots \quad (1.3)$$

and improves the previous result [11]: $R = 2$.

The spectral inequality (1.1) for the negative trace (that is, for $\gamma = 1$) is equivalent to the following integral inequality for orthonormal families. Let $\{\varphi_j\}_{j=1}^N \in H^1(\mathbb{R}^n)$ be an orthonormal family in $L_2(\mathbb{R}^n)$. Then $\rho(x) := \sum_{j=1}^N \varphi_j(x)^2$ satisfies the inequality

$$\int \rho(x)^{1+2/n} dx \leq k_n \sum_{j=1}^N \|\nabla \varphi_j\|^2, \quad (1.4)$$

where the best constants k_n and $L_{1,n}$ satisfy [20], [19]

$$k_n = (2/n)(1 + n/2)^{1+2/n} L_{1,n}^{2/n}. \quad (1.5)$$

In addition to the initial quantum mechanical applications inequality (1.4) is very important in the theory of infinite dimensional dynamical systems, especially, for the attractors of the Navier–Stokes equations (see, for instance, [19], [3], [6], [7], [23] and the references therein). Accordingly, for satisfying these needs Lieb–Thirring inequalities (1.4) were generalized to higher-order elliptic operators on domains with various boundary conditions and Riemannian manifolds [10], [23]. However, no information was available on the values of the corresponding constants. A different approach to the Lieb–Thirring inequalities for periodic functions, based on the methods of trigonometric series, was proposed in [16].

In this article we shall be dealing with Lieb–Thirring inequalities on manifolds. We consider the two-dimensional torus $T^2 = [0, 2\pi]^2$ (with flat metric) and the two-dimensional sphere S^2 . Below we denote by M either T^2 or S^2 . Both the scalar and vector-functions are considered. We first observe that for scalar functions inequality (1.4) cannot hold unless we somehow get rid of the constants, and we assume that the φ_j 's satisfy

$$\int_M \varphi dM = 0. \quad (1.6)$$

Accordingly, the Schrödinger operator is of the form

$$-\Delta \varphi + \Pi(V\varphi), \quad \text{where} \quad \Pi f = f - \frac{1}{|M|} \int_M f dM, \quad (1.7)$$

and $|M|$ denotes the measure of M . In section 2 we obtain a bound for the negative trace of the operator (1.7) on M

$$\sum_{\nu_j \leq 0} |\nu_j| \leq L_1(M) \int_M V_-(x)^2 dM \quad \text{with} \quad L_1(M) \leq \frac{3}{8}.$$

It is worth pointing out that we obtain the same bound as in the original paper [20] for the constant $L_{1,2}(\mathbb{R}^2)$. As in [20] we use the Birman–Schwinger kernel (see also [23]). The current best known results (1.3) for \mathbb{R}^n are, of course, much sharper. However, the argument in [2] and induction in the dimension [8], [18], [11] are not directly applicable to the case of the torus and the sphere because of the global condition (1.6) (especially since on the sphere there is no global coordinate system without singular points).

Next, we consider the case of vector-functions and show that

$$L_1^{\text{vec}}(M) \leq \frac{3}{4}. \quad (1.8)$$

This is, of course, obvious for the torus since the vector Laplacian acts independently on the two components of vector-functions. This is not the case for the sphere, but (1.8) still holds. We also observe that for the sphere (as for any simply connected manifold) we do not need any orthogonality conditions and the (negative) vector Laplacian is strictly positive on \mathbb{S}^2 . Using the one-to-one correspondence between divergence-free and potential vector fields inherent in two dimensions we show that in the divergence-free case the bound for the corresponding Lieb–Thirring constant is the same as in the scalar case. Finally, in the three-dimensional case we prove the inequality for the negative trace for \mathbb{T}^3 with the original Lieb–Thirring constant $\frac{4}{15\pi}$ [20] and some 1.039% larger constant for \mathbb{S}^3 .

In section 3 we consider the one-dimensional case. Using the idea of C. Foias [23, p. 440] (see also [9]) and a recent refinement [4] of the multiplicative inequality characterizing the imbedding $\dot{H}^1(\mathbb{S}^1) \hookrightarrow L_\infty(\mathbb{S}^1)$ we obtain for the operator

$$-\frac{d^2\varphi}{dx^2} + \Pi(V\varphi),$$

acting on 2π -periodic functions with mean value zero the following simultaneous bound for the negative trace and the number N of negative eigenvalues:

$$\sum_{j=1}^N |\nu_j| + N \frac{1}{\pi^2} \leq \frac{2}{3\sqrt{3}} \int_0^{2\pi} V(x)_-^{3/2} dx.$$

In section 4 we prove two main technical results concerning a series and a 2D lattice sum depending on a parameter. Corresponding to these sums in \mathbb{R}^n are the integrals depending on a parameter which

are easily calculated by scaling. The previous (knowingly non-sharp) estimates for these sums in [12], [15] give, respectively, $L_1(\mathbb{S}^2) \leq 1/2$ and $L_1(\mathbb{T}^2) \leq 3/(2\pi)$.

In conclusion we recall the basic facts concerning the Laplace operator on the sphere [21]. Let \mathbb{S}^{m-1} be the $(m-1)$ -dimensional sphere. We have for the (scalar) Laplace-Beltrami operator $\Delta = \text{div grad}$:

$$-\Delta Y_n^k = \Lambda_n Y_n^k, \quad k = 1, \dots, k_m(n), \quad n = 1, 2, \dots$$

Here the Y_n^k are the orthonormal spherical harmonics. Each eigenvalue

$$\Lambda_n = n(n + m - 2)$$

has multiplicity

$$k_m(n) = \frac{2n + m - 2}{n} \binom{n + m - 3}{n - 1}.$$

For example, for $m = 2, 3, 4$ we have

$$\begin{aligned} \mathbb{S}^1 : \quad \Lambda_n &= n^2, \quad k_2(n) = 2, \\ \mathbb{S}^2 : \quad \Lambda_n &= n(n + 1), \quad k_3(n) = 2n + 1, \\ \mathbb{S}^3 : \quad \Lambda_n &= n(n + 2), \quad k_4(n) = (n + 1)^2. \end{aligned} \tag{1.9}$$

The following identity is essential [21]: for any $s \in \mathbb{S}^{m-1}$

$$\sum_{l=1}^{k_m(n)} Y_n^l(s)^2 = \frac{k_m(n)}{\sigma(m)}, \tag{1.10}$$

where $\sigma(m) = 2\pi^{m/2}/\Gamma(m/2)$ is the surface area of S^{m-1} . In the vector case we have the similar identity for the gradients of spherical harmonics [13]: for any $s \in \mathbb{S}^{m-1}$

$$\sum_{l=1}^{k_m(n)} |\nabla Y_n^l(s)|^2 = \Lambda_n \frac{k_m(n)}{\sigma(m)}, \tag{1.11}$$

We also use the following notation labelling the eigenfunctions and the corresponding eigenvalues with a single subscript

$$-\Delta \varphi_i = \lambda_i \varphi_i, \tag{1.12}$$

where

$$\{\varphi_i\}_{i=1}^\infty = \{Y_n^1, \dots, Y_n^{k_m(n)}\}_{n=1}^\infty, \quad \{\lambda_i\}_{i=1}^\infty = \{\Lambda_n, \dots, \Lambda_n\}_{n=1}^\infty.$$

$k_m(n) \text{ times}$

2. LIEB–THIRRING INEQUALITIES ON THE SPHERE AND ON THE TORUS

In this section we obtain estimates for the negative trace of the Schrödinger operators on the $2D$ sphere \mathbb{S}^2 and the $2D$ torus $\mathbb{T}^2 = [0, 2\pi]^2$. Both cases are treated simultaneously and we denote below by M one of these manifolds. With a slight abuse of notation a generic point $x \in \mathbb{T}^2$ and $s \in \mathbb{S}^2$ is denoted by x .

For $V \in L_2(M)$ we consider the quadratic form on $\dot{H}^1(M)$

$$Q_V(h) = \|\nabla h\|^2 + \int_M V(x)h(x)^2 dM, \quad h \in \dot{H}^1(M). \quad (2.1)$$

Here and in what follows $\dot{H}^1(M)$ denotes the subspace of the Sobolev space $H^1(M)$ of functions orthogonal to constants. The form (2.1) is bounded from below and defines the self-adjoint Schrödinger-type operator

$$-\Delta h + \Pi(Vh), \quad h \in \dot{H}^1(M) \quad (2.2)$$

with discrete spectrum $\nu_1 \leq \nu_2 \leq \dots \rightarrow \infty$ accumulating at infinity.

We estimate the negative trace of (2.2) for $M = \mathbb{S}^2$ and $M = \mathbb{T}^2$

$$\sum_{\nu_j \leq 0} |\nu_j| \leq L_1(M) \int_M V_-(x)^2 dM. \quad (2.3)$$

Theorem 2.1. *For $M = \mathbb{S}^2$ and $M = \mathbb{T}^2$*

$$L_1(\mathbb{T}^2) < \frac{3}{8}, \quad L_1(\mathbb{S}^2) < \frac{3}{8}. \quad (2.4)$$

Proof. As usual we first assume that the potential V is smooth. Having proved (2.3) for smooth V we prove the general case by approximating V with smooth potentials V_n . We denote by $N_r(V)$ the number of eigenvalues ν_j such that $\nu_j \leq r$. Then

$$\sum_{\nu_j \leq 0} |\nu_j|^\gamma = \gamma \int_0^\infty r^{\gamma-1} N_{-r}(V) dr. \quad (2.5)$$

We use the Birman–Schwinger inequality (see [23, Appendix, Proposition 2.1], where this inequality is adapted to the Schrödinger-type operators defined on subspaces). Setting $g(x) = (V(x) + (1-t)r)_-$, we have

$$N_{-r}(V) \leq \text{Tr}[g^{1/2}(\Pi(-\Delta + tr)\Pi)^{-1}g^{1/2}]^k, \quad r > 0, \quad k \geq 1, \quad t \in [0, 1],$$

where the trace is calculated in $L_2(M)$. Next we use the convexity inequality of Lieb and Thirring [1], [20]: for positive operators A and C , $\text{Tr}(A^{1/2}CA^{1/2})^k \leq \text{Tr} A^{k/2}C^kA^{k/2}$. We obtain

$$N_{-r}(V) \leq \text{Tr}[g^{k/2}(\Pi(-\Delta + tr)\Pi)^{-k}g^{k/2}] = \text{Tr}[g^k(\Pi(-\Delta + tr)\Pi)^{-k}],$$

where the last equality holds for $k > 1$, since in this case the operator $(\Pi(-\Delta + tr)\Pi)^{-k}$ is of trace class (and multiplication by $g^{k/2}$ is bounded in $L_2(M)$).

Now we show that for $k > 1$ ($k = 3/2$),

$$N_{-r}(V) \leq \frac{1}{4\pi} \frac{1}{k-1} (tr)^{1-k} \int_M (V(x) + (1-t)r)_-^k dM. \quad (2.6)$$

We first consider the case $M = \mathbb{S}^2$. Using the basis (1.12) and identity (1.10), we have

$$\begin{aligned} \text{Tr}[g^k(\Pi(-\Delta + tr)\Pi)^{-k}] &= \sum_{j=1}^{\infty} (g^k(-\Delta + tr)^{-k} \varphi_j, \varphi_j) \\ &= \sum_{j=1}^{\infty} (\lambda_j + tr)^{-k} \int_{\mathbb{S}^2} g(s)^k \varphi_j(s)^2 dS \\ &= \sum_{n=1}^{\infty} (\Lambda_n + tr)^{-k} \int_{\mathbb{S}^2} g(s)^k \sum_{l=1}^{2n+1} (Y_n^l(s))^2 dS \\ &= \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1) + tr)^k} \int_{\mathbb{S}^2} g(s)^k dS, \end{aligned}$$

which proves (2.6) for $M = \mathbb{S}^2$ in view of Proposition 4.1.

For the torus \mathbb{T}^2 we use the orthonormal basis $(2\pi)^{-1}e^{imx}$, $m \in \mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus 0$ and obtain

$$N_{-r}(V) \leq \frac{1}{4\pi^2} \sum_{m \in \mathbb{Z}_0^2} \frac{1}{(|m|^2 + tr)^k} \int_{\mathbb{T}^2} g(x)^k dx,$$

which proves (2.6) for $M = \mathbb{T}^2$ in view of Proposition 4.2.

Next, restricting k to $k \in (1, 2)$ and using (2.5) with $\gamma = 1$ we have

$$\sum_{\nu_j \leq 0} |\nu_j| \leq \frac{1}{4\pi} \frac{1}{k-1} \int_M \int_0^\infty (tr)^{1-k} (V(x) + (1-t)r)_-^k dr dx.$$

We evaluate the inner integral setting $r = \frac{1}{1-t}V_-(x)\rho$. If $V \leq 0$ and $V_- = -V$, then $(V(x) + (1-t)r)_- = V_-(x)(\rho - 1)_-$ and

$$\int_0^\infty (tr)^{1-k}(V(x) + (1-t)r)_-^k dr = t^{1-k}(1-t)^{k-2}B(2-k, 1+k)V_-(x)^2.$$

For the optimal $t = k - 1 \in (0, 1)$ we obtain

$$\sum_{\nu_j \leq 0} |\nu_j| \leq \frac{1}{4\pi} \frac{1}{k-1} \frac{B(2-k, 1+k)}{(k-1)^{k-1}(2-k)^{2-k}} \int_M V_-(x)^2 dM, \quad k \in (1, 2), \quad (2.7)$$

which proves (2.3) with

$$L_1(M) \leq \frac{1}{4\pi} \frac{B(2-k, 1+k)}{(k-1)^k(2-k)^{2-k}} \Big|_{k=3/2} = \frac{3}{8}.$$

The minimum is attained at $k = 1.38\dots$, giving $L_1(M) \leq 0.3605\dots$. \square

We now consider the vector case important for applications. The case $M = \mathbb{T}^2$ involves no difficulties since the Laplacian acts independently on the components of a vector field, so we consider $M = \mathbb{S}^2$. The Laplace operator acting on (tangent) vector fields on \mathbb{S}^2 we define as the Laplace-de Rham operator $-d\delta - \delta d$ identifying 1-forms and vectors. Then for a two-dimensional manifold we have [13]

$$\Delta u = \nabla \operatorname{div} u - \operatorname{rot} \operatorname{rot} u,$$

where the operators $\nabla = \operatorname{grad}$ and div have the conventional meaning. The operator rot of a vector u is a scalar and for a scalar ψ , $\operatorname{rot} \psi$ is a vector:

$$\operatorname{rot} u := -\operatorname{div}(n \times u), \quad \operatorname{rot} \psi := -n \times \nabla \psi,$$

where n is the unit outward normal vector. We note that for the operators rot so defined, for a scalar ψ it holds

$$\operatorname{rot} \operatorname{rot} \psi = -\Delta \psi (= -\operatorname{div} \operatorname{grad} \psi). \quad (2.8)$$

Integrating by parts, that is, using

$$(\nabla \psi, u)_{L_2(T\mathbb{S}^2)} = -(\psi, \operatorname{div} u)_{L_2(\mathbb{S}^2)}, \quad (\operatorname{rot} \psi, u)_{L_2(T\mathbb{S}^2)} = (\psi, \operatorname{rot} u)_{L_2(\mathbb{S}^2)},$$

we obtain

$$(-\Delta u, u)_{L_2(T\mathbb{S}^2)} = \|\operatorname{rot} u\|^2 + \|\operatorname{div} u\|^2.$$

Next, we have the orthogonal sum $L_2(T\mathbb{S}^2) = H \oplus H^\perp$:

$$H = \{u \in L_2(T\mathbb{S}^2), \operatorname{div} u = 0\}, \quad H^\perp = \{u \in L_2(T\mathbb{S}^2), \operatorname{rot} u = 0\}.$$

Both H and H^\perp are invariant with respect to Δ (in then sense that if $u \in H$ and $\Delta u \in L_2(T\mathbb{S}^2)$, then $\Delta u \in H$, and similarly for H^\perp) and there exist two orthonormal systems of eigenvectors: $\{w_j\}_{j=1}^\infty \in H$ and $\{v_j\}_{j=1}^\infty \in H^\perp$ with the same eigenvalues

$$-\Delta w_j = \lambda_j w_j, \quad -\Delta v_j = \lambda_j v_j, \quad (2.9)$$

where

$$w_j = \lambda_j^{-1/2} n \times \nabla \varphi_j, \quad v_j = \lambda_j^{-1/2} \nabla \varphi_j.$$

Here the λ_j 's and the φ_j 's are the eigenvalues and eigenfunctions of the scalar Laplacian on \mathbb{S}^2 , see (1.12). Both (2.9), and the orthonormality of the w_j 's and v_j 's follow from (2.8). Hence, corresponding to the eigenvalue $\Lambda_n = n(n+1)$ there are two families of $2n+1$ orthonormal eigenvectors $w_n^l(s)$ and $v_n^l(s)$, $l = 1, \dots, 2n+1$ and (1.11) gives the following important identities: for any $s \in \mathbb{S}^2$

$$\sum_{l=1}^{2n+1} |w_n^l(s)|^2 = \frac{2n+1}{4\pi}, \quad \sum_{l=1}^{2n+1} |v_n^l(s)|^2 = \frac{2n+1}{4\pi}. \quad (2.10)$$

We finally observe that $-\Delta \geq \Lambda_1 I = 2I$.

Having done these preliminaries we consider the quadratic form

$$Q_V^{\text{vec}}(u) = \|\text{rot } u\|^2 + \|\text{div } u\|^2 + \int_{\mathbb{S}^2} V(s)|u(s)|^2 dS, \quad u \in H^1(T\mathbb{S}^2), \quad (2.11)$$

which is bounded from below, and defines the self-adjoint Schrödinger operator

$$-\Delta u + Vu$$

with discrete spectrum. We estimate its negative trace

$$\sum_{\nu_j \leq 0} |\nu_j| \leq L_1^{\text{vec}}(\mathbb{S}^2) \int_{\mathbb{S}^2} V_-(s)^2 dS. \quad (2.12)$$

Theorem 2.2.

$$L_1^{\text{vec}}(\mathbb{S}^2) \leq \frac{3}{4}. \quad (2.13)$$

Proof. Using the basis (2.9), identity (2.10), similarly to Theorem 2.1

$$\begin{aligned} N_{-r}(V) &\leq \text{Tr}[g^k(-\Delta + tr)]^{-k} \\ &= \sum_{j=1}^{\infty} (g^k(-\Delta + tr)^{-k} w_j, w_j) + \sum_{j=1}^{\infty} (g^k(-\Delta + tr)^{-k} v_j, v_j) \\ &= 2 \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1) + tr)^k} \int_{\mathbb{S}^2} g(s)^k dS \leq \frac{1}{2\pi} \frac{1}{k-1} (tr)^{1-k} \int_{\mathbb{S}^2} g(s)^k dS, \end{aligned}$$

and we complete the proof as in Theorem 2.1. \square

Remark 2.1. The same estimate holds for the torus

$$L_1^{\text{vec}}(\mathbb{T}^2) \leq \frac{3}{4}. \quad (2.14)$$

However, in this case we have to assume that u has zero average.

Spectral inequalities (2.3) and (2.12) are equivalent to the integral inequalities for families of orthonormal functions and vector fields. As before, M stands for \mathbb{S}^2 or \mathbb{T}^2 .

Theorem 2.3. *Let $\{\varphi_j\}_{j=1}^N \in \dot{H}^1(M)$ be an orthonormal scalar family. Then for $\rho(x) := \sum_{j=1}^N \varphi_j(x)^2$ the following inequality holds:*

$$\int_M \rho(x)^2 dM \leq k_2 \sum_{j=1}^N \|\nabla \varphi_j\|^2, \quad k_2 \leq \frac{3}{2}. \quad (2.15)$$

If a family of vector fields $\{u_j\}_{j=1}^N \in H^1(TM)$ is orthonormal in $L^2(TM)$, then

$$\int_M \rho(x)^2 dM \leq k_2^{\text{vec}} \sum_{j=1}^N (\|\text{rot } u_j\|^2 + \|\text{div } u_j\|^2), \quad k_2^{\text{vec}} \leq 3, \quad (2.16)$$

where $\rho(x) = \sum_{j=1}^N |u_j(x)|^2$. If, in addition, $\text{div } u_j = 0$ (or $\text{rot } u_j = 0$) for $j = 1, \dots, N$, then

$$\int_M \rho(x)^2 dM \leq \begin{cases} k_2^{\text{sol}} \sum_{j=1}^N \|\text{rot } u_j\|^2, & \text{div } u_j = 0, \\ k_2^{\text{pot}} \sum_{j=1}^N \|\text{div } u_j\|^2, & \text{rot } u_j = 0, \end{cases} \quad (2.17)$$

where

$$k_2^{\text{sol}} = k_2^{\text{pot}} \leq \frac{k_2^{\text{vec}}}{2} \leq \frac{3}{2}. \quad (2.18)$$

Proof. In two dimensions the relation (1.5) between the constants k_2 and L_1 is as follows (the fact that we are dealing with manifolds does not play a role)

$$k_2 = 4L_1. \quad (2.19)$$

This proves (2.15) and (2.16). For the sake of completeness we recall the proof of (2.17), (2.18) from [15]. By symmetry inherent in the two-dimensional case

$$\operatorname{div} u = 0 \Leftrightarrow \operatorname{rot} \widehat{u} = 0, \quad \text{where} \quad \widehat{u} = n \times u.$$

Furthermore, u_1, \dots, u_N are orthonormal if and only if $\widehat{u}_1, \dots, \widehat{u}_N$ are orthonormal. This shows that $k_2^{\text{sol}} = k_2^{\text{pot}}$. Let us prove the inequality $k_2^{\text{sol}} \leq \frac{k_2^{\text{vec}}}{2}$. Let u_1, \dots, u_N be orthonormal and let $\operatorname{div} u_j = 0$, $j = 1, \dots, N$. We set $\rho(x) = \sum_{j=1}^N |u_j(x)|^2$ and consider the family of $2N$ vector functions $u_1, \dots, u_N, \widehat{u}_1, \dots, \widehat{u}_N$. Since $\operatorname{div} u_j = 0$ and $\operatorname{rot} \widehat{u}_j = 0$, $j = 1, \dots, N$, we have $(u_i, \widehat{u}_j) = 0$ for $1 \leq i, j \leq N$, and the whole family is orthonormal. Applying (2.16) to this family of $2N$ functions and taking into account that $|u_j(x)| = |\widehat{u}_j(x)|$ and $\operatorname{div} \widehat{u}_j(x) = -\operatorname{rot} u_j(x)$ we obtain

$$\begin{aligned} 4 \int_M \rho(x)^2 dx &= \int_M \left(\sum_{j=1}^N (|u_j(x)|^2 + |\widehat{u}_j(x)|^2) \right)^2 dx \leq \\ &\leq k_2^{\text{vec}} \sum_{j=1}^N (\|\operatorname{rot} u_j\|^2 + \|\operatorname{div} \widehat{u}_j\|^2) = 2k_2^{\text{vec}} \sum_{j=1}^N \|\operatorname{rot} u_j\|^2. \end{aligned}$$

Therefore $k_2^{\text{sol}} \leq k_2^{\text{vec}}/2 \leq 3/2$. \square

Remark 2.2. The lower bound for $k_2(M)$ is the same as in \mathbb{R}^2

$$k_2(M) \geq \frac{1}{2\pi}. \quad (2.20)$$

For instance, for the sphere we take the first N eigenfunctions (1.12) and use the fact that $\lambda_j = [j^{1/2}][j^{1/2}] + 1 \sim j$. Then

$$N^2 = \left(\int_{\mathbb{S}^2} \rho(s) dS \right)^2 \leq 4\pi \|\rho\|^2 \leq 4\pi k_2 \sum_{j=1}^N \lambda_j \sim 2\pi k_2 N^2.$$

Accordingly, in view of (2.19),

$$L_1(M) \geq \frac{1}{8\pi}.$$

The same lower bound holds for \mathbb{T}^2 since in this case $\lambda_j \sim j/\pi$.

Concluding this section we briefly consider the three-dimensional case. For \mathbb{S}^3 we see from (1.9) that the eigenvalue $\Lambda_n = n(n+2)$ has multiplicity $(n+1)^2$ and arguing as in Theorem 2.1 and setting $k = 2$ we obtain using Proposition 4.3

$$\begin{aligned} N_{-r}(V) &\leq \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n(n+2) + tr)^2} \int_{\mathbb{S}^3} g(s)^2 dS \\ &\leq \frac{\delta_{\mathbb{S}^3}}{8\pi} (tr)^{-1/2} \int_{\mathbb{S}^3} g(s)^2 dS. \end{aligned}$$

For the torus \mathbb{T}^3 using the basis of exponentials $(2\pi)^{-3/2} e^{imx}$, $m \in \mathbb{Z}_0^3$ we have

$$\begin{aligned} N_{-r}(V) &\leq \frac{1}{8\pi^3} \sum_{m \in \mathbb{Z}_0^3} \frac{1}{(|m|^2 + tr)^2} \int_{\mathbb{T}^2} g(x)^2 dx \\ &< \frac{\delta_{\mathbb{T}^3}}{8\pi} (tr)^{-1/2} \int_{\mathbb{T}^3} g(x)^2 dx. \end{aligned}$$

We set $t = 1/2$ and for a fixed $x \in M$ calculate the integral

$$\int_0^\infty (tr)^{-1/2} (V(x) + (1-t)r)_-^2 dr = \frac{32}{15} V_-(x)^{5/2}$$

and obtain using (2.5) the following result.

Theorem 2.4. *The negative spectrum of the operator $-\Delta + \Pi(V \cdot)$ on $M = \mathbb{S}^3$ or \mathbb{T}^3 satisfies*

$$\sum_{\nu_j \leq 0} |\nu_j| \leq L_1(M) \int_M V_-(x)^{5/2} dM,$$

where

$$L_1(M) \leq \delta_M \frac{4}{15\pi}.$$

Here $\delta_{\mathbb{S}^3} = 1.0139 \dots$ and $\delta_{\mathbb{T}^3} = 1$.

3. ONE-DIMENSIONAL TWO-TERM LIEB-THIRRING INEQUALITIES

The imbedding of the Sobolev space $H^l(\mathbb{R})$, $l > 1/2$, into the space of bounded continuous functions can be written in the form of a multiplicative inequality

$$\|f\|_\infty^2 \leq c(l) \|f\|^{2-1/l} \|f^{(l)}\|^{1/l}, \quad (3.1)$$

where the sharp constant $c(l)$ was found in [22]:

$$c(l) = (2l\alpha^\alpha (1-\alpha)^{1-\alpha} \sin \pi\alpha)^{-1}, \quad \alpha = 1/(2l). \quad (3.2)$$

It was also shown there that there exists a unique (up to dilations and translations) extremal function. For periodic functions with zero average $f \in \dot{H}^l(\mathbb{S}^1)$ inequality (3.1) holds with the same sharp constant (3.2), however, there are no extremal functions [14]. An important improvement of (3.1) for 2π -periodic functions has been recently obtained in [4], where it was shown that

$$\|f\|_\infty^2 \leq c(l)\|f\|^{2-1/l}\|f^{(l)}\|^{1/l} - K(l)\|f\|^2. \quad (3.3)$$

For all l the constant $K(l) > 0$ and, in particular, $K(1) = 1/\pi$ and $K(2) = 2/(3\pi)$, so that

$$\|f\|_\infty^2 \leq 1 \cdot \|f\| \|f'\| - \frac{1}{\pi} \|f\|^2, \quad \|f\|_\infty^2 \leq (4/27)^{1/4} \|f\|^{3/2} \|f''\|^{1/2} - \frac{2}{3\pi} \|f\|^2, \quad (3.4)$$

where all four constants are sharp and no extremal functions exist.

Theorem 3.1. *Suppose that $\{\varphi_j\}_{j=1}^N \subset \dot{H}^l(\mathbb{S}^1)$ is an orthonormal family in $L_2(\mathbb{S}^1)$. Then for $\rho(x) := \sum_{j=1}^N \varphi_j(x)^2$ the following inequality holds:*

$$\int_0^{2\pi} \rho(x)^{2l+1} dx + N \cdot K(l)^{2l} \leq c(l)^{2l} \sum_{j=1}^N \|\varphi_j^{(l)}\|^2. \quad (3.5)$$

Proof. For any $\xi \in \mathbb{R}^N$ using (3.3) with $f(x) = \sum_{j=1}^N \xi_j \varphi_j(x)$ we have

$$\left| \sum_{j=1}^N \xi_j \varphi_j(x) \right|^2 \leq c(l) \left(\sum_{j=1}^N \xi_j^2 \right)^{\frac{2l-1}{2l}} \left(\sum_{i,j=1}^N \xi_i \xi_j (\varphi_i^{(l)}, \varphi_j^{(l)}) \right)^{\frac{1}{2l}} - K(l) \left(\sum_{j=1}^N \xi_j^2 \right),$$

by orthonormality. Setting $\xi_j = \varphi_j(x)$ we obtain

$$\rho(x)^2 \leq c(l) \rho(x)^{\frac{2l-1}{2l}} \left(\sum_{i,j=1}^N \varphi_i(x) \varphi_j(x) (\varphi_i^{(l)}, \varphi_j^{(l)}) \right)^{\frac{1}{2l}} - K(l) \rho(x),$$

or

$$\begin{aligned} \rho(x)^{2l+1} + K(l)^{2l} \rho(x) &\leq \rho(x) (\rho(x) + K(l))^{2l} \\ &\leq c(l)^{2l} \sum_{i,j=1}^N \varphi_i(x) \varphi_j(x) (\varphi_i^{(l)}, \varphi_j^{(l)}). \end{aligned}$$

Integrating and again using orthonormality we finally obtain (3.5). \square

For $V(x) \geq 0$ we consider the following quadratic form on $\dot{H}^l(\mathbb{S}^1)$

$$\int_0^{2\pi} \varphi^{(l)}(x)^2 dx - \int_0^{2\pi} V(x) \varphi(x)^2 dx, \quad (3.6)$$

which is bounded from below and defines a Schrödinger-type operator

$$-\frac{d^{2l}\varphi}{dx^{2l}} - \Pi(V\varphi). \quad (3.7)$$

In view of compactness of \mathbb{S}^1 the spectrum of this operator is discrete.

Theorem 3.2. *Suppose that there exist N negative eigenvalues $-\nu_j \leq 0$, $j = 1, \dots, N$ of the operator (3.7). Then both the negative trace and the number N of negative eigenvalues satisfy the following inequality*

$$\sum_{j=1}^N \nu_j + N \cdot \left(\frac{K(l)}{c(l)} \right)^{2l} \leq \frac{2l}{(2l+1)^{\frac{2l+1}{2l}}} \cdot c(l) \int_0^{2\pi} V(x)^{\frac{2l+1}{2l}} dx. \quad (3.8)$$

Proof. Let the orthonormal eigenfunctions $\varphi_j(x)$ correspond to the eigenvalues $-\nu_j$. Then

$$\int_0^{2\pi} \varphi_j^{(l)}(x)^2 dx - \int_0^{2\pi} V(x) \varphi_j(x)^2 dx = -\nu_j.$$

Setting as before $\rho(x) := \sum_{j=1}^N \varphi_j(x)^2$ and using (3.5) we obtain

$$\begin{aligned} \sum_{j=1}^N \nu_j &= \int_0^{2\pi} V(x) \rho(x) dx - \sum_{j=1}^N \|\varphi_j^{(l)}\|^2 \\ &\leq \|V\|_{L^{\frac{2l+1}{2l}}} \|\rho\|_{L^{2l+1}} - \frac{1}{c(l)^{2l}} \|\rho\|_{L^{2l+1}}^{2l+1} - N \cdot \left(\frac{K(l)}{c(l)} \right)^{2l} \\ &\leq \max_y \left(\|V\|_{L^{\frac{2l+1}{2l}}} y - \frac{1}{c(l)^{2l}} y^{2l+1} \right) - N \cdot \left(\frac{K(l)}{c(l)} \right)^{2l}. \end{aligned}$$

Calculating the maximum we obtain (3.8). \square

Remark 3.1. It is worth pointing out that unlike $c(l)$, the constants $K(l)$ are not dimensionless and for L -periodic functions (with mean

value zero) we have $K_L(l) = K_{2\pi}(l)(2\pi/L)$. For example, for $l = 1$

$$\begin{aligned} \int_0^L \rho(x)^3 dx + N \frac{4}{L^2} &\leq \sum_{j=1}^N \|\varphi_j'\|^2, \\ \sum_{j=1}^N \nu_j + N \frac{4}{L^2} &\leq \frac{2}{3\sqrt{3}} \int_0^L V(x)^{3/2} dx. \end{aligned} \tag{3.9}$$

Remark 3.2. If the potential V is even (and periodic), then the subspace of odd periodic functions is invariant for the operator

$$-\frac{d^{2l}}{dx^{2l}}\varphi - V\varphi,$$

and the orthogonal projection Π (1.7) can be omitted.

4. AUXILIARY INEQUALITIES

Proposition 4.1. *For $\mu \geq 0$ and $k = 3/2$*

$$H(\mu) := \mu^{2k-2} \sum_{n=1}^{\infty} \frac{2n+1}{((n(n+1) + \mu^2)^k)} < \frac{1}{k-1}. \tag{4.10}$$

Proof. Since

$$H(\mu) = \mu^{-2} \sum_{n=1}^{\infty} (2n+1) f(n(n+1)/\mu^2), \tag{4.11}$$

where

$$f(x) = \frac{1}{(x+1)^k} \quad \text{and} \quad \int_0^{\infty} f(x) dx = \frac{1}{k-1},$$

the fact that inequality (4.10) holds for all $\mu \geq \mu_0$, where μ_0 is sufficiently large, follows from Lemma 4.1 below, which gives the asymptotic expansion of $H(\mu)$ for large μ :

$$H(\mu) = \frac{1}{k-1} - \frac{2}{3} \frac{1}{\mu^2} + o(1/\mu^2).$$

The point $\mu_0 = 5.0833$ is specified in the Appendix (see section 5). On the *finite* interval $[0, \mu_0]$ we make sure that (4.10) holds by numerical calculations. The graph of $H(\mu)$ on $[0, \mu_0]$ is shown in Fig. 1. \square

Lemma 4.1. *Suppose that f is sufficiently smooth and sufficiently fast decays at infinity. Then the following asymptotic expansion as $\mu \rightarrow \infty$ holds for $H(\mu)$ defined in (4.11):*

$$H(\mu) = \int_0^\infty f(x)dx - \frac{1}{\mu^2} \frac{2}{3} f(0) + o(1/\mu^2). \quad (4.12)$$

Proof. We consider the following partitioning of the half-line $x \geq 0$ by the points

$$a_n = a_n(\mu) = \frac{(n-1)n}{\mu^2}, \quad n = 1, \dots$$

Then a direct inspection shows that

$$\begin{aligned} \mu^{-2} \sum_{n=1}^{\infty} n f(n(n+1)/\mu^2) &= \frac{1}{2} \sum_{n=1}^{\infty} f(a_{n+1})(a_{n+1} - a_n), \\ \mu^{-2} \sum_{n=1}^{\infty} (n+1) f(n(n+1)/\mu^2) &= \frac{1}{2} \sum_{n=1}^{\infty} f(a_{n+1})(a_{n+2} - a_{n+1}). \end{aligned}$$

Therefore

$$H(\mu) = \frac{1}{2} f(a_2)(a_2 - a_1) + \sum_{n=2}^{\infty} \frac{f(a_n) + f(a_{n+1})}{2} (a_{n+1} - a_n).$$

Next, we recall the trapezoidal formula for the approximate calculation of the integrals (see, for instance, [17]):

$$\int_a^b f(x)dx = \frac{f(a) + f(b)}{2} (b - a) + R_{a,b}(f), \quad (4.13)$$

where

$$R_{a,b}(f) = -\frac{(b-a)^3}{12} f''(\xi), \quad a < \xi < b.$$

This gives

$$\begin{aligned} \int_0^\infty f(x)dx &= \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx \\ &= \int_{a_1}^{a_2} f(x)dx + \sum_{n=2}^{\infty} \frac{f(a_n) + f(a_{n+1})}{2} (a_{n+1} - a_n) + \sum_{n=2}^{\infty} R_{a_n, a_{n+1}}(f) \\ &= H(\mu) + \int_{a_1}^{a_2} (f(x) - f(a_2)/2)dx + \sum_{n=2}^{\infty} R_{a_n, a_{n+1}}(f). \end{aligned} \quad (4.14)$$

Since $a_1 = 0$ and $a_2 = 2/\mu^2$ we clearly have

$$\lim_{\mu \rightarrow \infty} \mu^2 \int_{a_1}^{a_2} (f(x) - f(a_2)/2) dx = f(0).$$

For the third term, using (4.13) with

$$\xi_n \in (a_n, a_{n+1}), \quad \xi_n = \frac{n^2}{\mu^2} + \frac{\theta_n n}{\mu^2}, \quad |\theta_n| < 1 \quad (4.15)$$

we obtain

$$\begin{aligned} \lim_{\mu \rightarrow \infty} \mu^2 \sum_{n=2}^{\infty} R_{a_n, a_{n+1}}(f) &= -\frac{2}{3} \lim_{\mu \rightarrow \infty} \frac{1}{\mu} \sum_{n=1}^{\infty} (n/\mu)^3 f''(\xi_n) \\ &= -\frac{2}{3} \lim_{\mu \rightarrow \infty} \frac{1}{\mu} \sum_{n=1}^{\infty} (n/\mu)^3 f''(n^2/\mu^2) = -\frac{2}{3} \int_0^{\infty} x^3 f''(x^2) dx = -\frac{1}{3} f(0), \end{aligned}$$

as the following integration by parts shows:

$$\int_0^{\infty} x^3 f''(x^2) dx = \frac{1}{2} \int_0^{\infty} x^2 [f'(x^2)]'_x dx = - \int_0^{\infty} x f'(x^2) dx = \frac{1}{2} f(0).$$

Thus, the last two terms in (4.14) are both of order $1/\mu^2$ and add up to $\frac{2}{3\mu^2} f(0)$. The proof is complete. \square

Proposition 4.2. *For $\mu \geq 0$ and $k = 3/2$*

$$F(\mu) := \mu^{2k-2} \sum_{m \in \mathbb{Z}_0^2} \frac{1}{(|m|^2 + \mu^2)^k} < \frac{\pi}{k-1}. \quad (4.16)$$

Proof. The function $F(\mu)$ for $k > 1$ has the following asymptotic expansion as $\mu \rightarrow \infty$:

$$F(\mu) = \frac{\pi}{k-1} - \frac{1}{\mu^2} + O(e^{-C\mu}). \quad (4.17)$$

This follows from the Poisson summation formula (see, e. g., [21])

$$\sum_{m \in \mathbb{Z}^n} f(m/\mu) = (2\pi)^{n/2} \mu^n \sum_{m \in \mathbb{Z}^n} \widehat{f}(2\pi m \mu), \quad (4.18)$$

where $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{i\xi x} dx$. For the function $f(x) = 1/(1+x^2)^{-k}$, $x \in \mathbb{R}^2$, this gives

$$F(\mu) = \frac{1}{\mu^2} \sum_{m \in \mathbb{Z}^2} f(m/\mu) - \frac{1}{\mu^2} f(0) = \frac{\pi}{k-1} - \frac{1}{\mu^2} + 2\pi \sum_{m \in \mathbb{Z}_0^2} \widehat{f}(2\pi \mu m).$$

The third term is exponentially small as $\mu \rightarrow \infty$ since f is analytic in the strip $\operatorname{Re} z_1 < a$, $\operatorname{Re} z_2 < a$, $a < \sqrt{2}/2$, and therefore $|\widehat{f}(\xi)| \leq C(a, k)e^{-a|\xi|}$, see Remark 4.1. This proves (4.17). Hence (4.16) holds for all $\mu \in [\mu_0, \infty)$.

To specify μ_0 for $k = 3/2$ we take advantage of the formula [21]:

$$\mathcal{F}(1/(1+x^2)^{(n+1)/2})(\xi) = \frac{1}{c_n(2\pi)^{n/2}}e^{-|\xi|}, \quad x \in \mathbb{R}^n, \quad (4.19)$$

where

$$\frac{1}{c_n} = \frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)} = \int_{\mathbb{R}^n} \frac{dx}{(1+x^2)^{(n+1)/2}}.$$

In the two-dimensional case with $k = 3/2$

$$F(\mu) = \frac{\pi}{k-1} - \frac{1}{\mu^2} + 2\pi \sum_{m \in \mathbb{Z}_0^2} e^{-2\pi\mu|m|}.$$

Therefore (4.16) is equivalent to showing that the inequality

$$2\pi \sum_{m \in \mathbb{Z}_0^2} e^{-2\pi\mu|m|} < \frac{1}{\mu^2} \quad (4.20)$$

holds for all $\mu > 0$. To estimate the series on the right-hand side we write down the numbers $|m|^2 = m_1^2 + m_2^2$, $m \in \mathbb{Z}_0^2$, in the increasing order counting multiplicities and denote them by λ_j : $\{\lambda_j\}_{j=1}^\infty = \{m_1^2 + m_2^2, m \in \mathbb{Z}_0^2\}$. For $\lambda \geq 1$ we denote by $N(\lambda)$ the number of λ_j 's less than or equal to λ (the number of points with integer coordinates inside the circle of radius $\sqrt{\lambda}$):

$$N(\lambda) = \sum_{\lambda_j \leq \lambda} 1.$$

We inscribe the circle of radius $\sqrt{\lambda}$ into the square with side $2\sqrt{\lambda} + 1$ and cross out the origin. We obtain

$$N(\lambda) \leq (2\sqrt{\lambda} + 1)^2 - 1 = 4\lambda + 4\sqrt{\lambda} \leq 8\lambda.$$

For $\lambda = \lambda_j$ this gives $j = N(\lambda_j) \leq 8\lambda_j$ so that $\lambda_j \geq \frac{j}{8}$.

Returning to (4.20) and setting below $L := \pi\mu/2\sqrt{2}$ we have

$$\begin{aligned} \sum_{m \in \mathbb{Z}_0^2} e^{-2\pi\mu|m|} &= \sum_{j=1}^{\infty} e^{-2\pi\mu\lambda_j^{1/2}} \leq \sum_{j=1}^{\infty} e^{-2Lj^{1/2}} = e^{-L} \sum_{j=1}^{\infty} e^{-L(2j^{1/2}-1)} \\ &\leq e^{-L} \sum_{j=1}^{\infty} e^{-Lj^{1/2}} < e^{-L} \int_0^{\infty} e^{-Lx^{1/2}} dx = \frac{2e^{-L}}{L^2} = \frac{16}{\pi^2\mu^2} e^{-\frac{\pi\mu}{2\sqrt{2}}}, \end{aligned}$$

and inequality (4.20) is satisfied for all $\mu \geq \mu_0 = \frac{2\sqrt{2}}{\pi} \log \frac{32}{\pi} = 2.0896$. In fact, $\lambda_j \geq j/4$ (see [15]), which gives $\mu \geq \mu_0 = \frac{2}{\pi} \log \frac{16}{\pi} = 1.0363$. On the *finite* interval $[0, \mu_0]$ we verify (4.16) on a computer, see Fig. 1. \square

Remark 4.1. Shifting for x_1 and x_2 the domain of integration by $\pm ia$ and using analyticity we obtain

$$|\widehat{f}(\xi)| \leq \frac{e^{-a|\xi|}}{2(k-1)(1-2a^2)^{k-1}},$$

and we can specify μ_0 for any fixed $k > 1$ similarly to $k = 3/2$.

Remark 4.2. Inequalities (4.10) and (4.16) hold for $k = 1.38\dots$

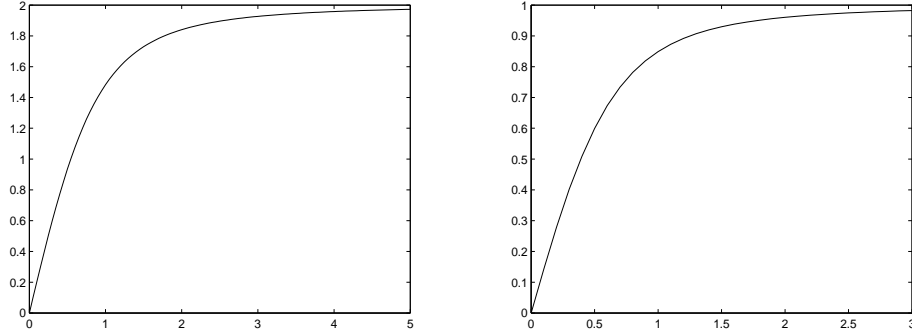


FIGURE 1. Graphs of the functions $H(\mu)$ and $\frac{k-1}{\pi}F(\mu)$ on the corresponding intervals $[0, \mu_0]$ for $k = 3/2$.

Proposition 4.3. *The following inequalities hold for $\mu \geq 0$:*

$$\begin{aligned} H_{\mathbb{S}^3}(\mu) &:= \mu \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n(n+2) + \mu^2)^2} \leq \delta_{\mathbb{S}^3} \int_0^{\infty} \frac{r^2 dr}{(r^2 + 1)^2} = \delta_{\mathbb{S}^3} \cdot \frac{\pi}{4}, \\ F_{\mathbb{T}^3}(\mu) &:= \mu \sum_{m \in \mathbb{Z}_0^3} \frac{1}{(|m|^2 + \mu^2)^2} < \delta_{\mathbb{T}^3} \int_{\mathbb{R}^3} \frac{dx}{(x^2 + 1)^2} = \delta_{\mathbb{T}^3} \cdot \pi^2, \end{aligned} \quad (4.21)$$

where $\delta_{\mathbb{S}^3} = 1.0139 \dots$ and $\delta_{\mathbb{T}^3} = 1$.

Proof. Calculations show that the function $H_{\mathbb{S}^3}(\mu)$ attains a global maximum at $\mu_* = 3.312 \dots$, which is $1.0139 \dots =: \delta_{\mathbb{S}^3}$ times greater than $H_{\mathbb{S}^3}(\infty) = \pi/4$. In calculations we can also take advantage of the fact that for $H_{\mathbb{S}^3}(\mu)$ there exists an explicit formula. In fact, using the formula

$$\sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \nu^2)^2} = \frac{\pi \coth(\pi \nu)}{4 \nu} + \frac{\pi^2}{4} (1 - \coth^2(\pi \nu)),$$

and noting that $n(n+2) = (n+1)^2 - 1$ we see that $H_{\mathbb{S}^3}(\mu)$ is equal to

$$\frac{\pi}{4} \frac{\mu}{\sqrt{\mu^2 - 1}} \coth(\pi \sqrt{\mu^2 - 1}) + \frac{\pi^2 \mu}{4} \left(1 - \coth^2(\pi \sqrt{\mu^2 - 1}) \right) - \frac{1}{\mu^3}.$$

Unlike the 2D case, for large μ , $H_{\mathbb{S}^3}(\mu) > H_{\mathbb{S}^3}(\infty) = \pi/4$.

For the second sum the Poisson summation formula and (4.19) give

$$F_{\mathbb{T}^3}(\mu) = \pi^2 - \frac{1}{\mu^3} + \pi^2 \sum_{m \in \mathbb{Z}_0^3} e^{-2\pi\mu|m|} = \pi^2 - \frac{1}{\mu^3} + O(e^{-C\mu}).$$

We find a μ_0 such that $F_{\mathbb{T}^3}(\mu) < \pi^2$ on $[\mu_0, \infty)$ and then verify the inequality on the remaining finite interval $[0, \mu_0]$ by calculations. We omit the details concerning μ_0 that are similar to those in Proposition 4.2. The graphs of $H_{\mathbb{S}^3}(\mu)$ and $F_{\mathbb{T}^3}(\mu)$ are shown in Fig. 2. \square

5. APPENDIX. ESTIMATE OF μ_0 FOR THE SPHERE

Lemma 5.1. *For $k = 3/2$ inequality (4.10) holds for $\mu \in [\mu_0, \infty)$, where $\mu_0 = 5.0833$.*

Proof. It follows from (4.14) that we have to show that for $f(x) = 1/(x+1)^k$ and $\mu \geq \mu_0$

$$\int_{a_1}^{a_2} f(x) dx - a_2 f(a_2)/2 > - \sum_{n=2}^{\infty} R_{a_n, a_{n+1}}(f), \quad (5.1)$$

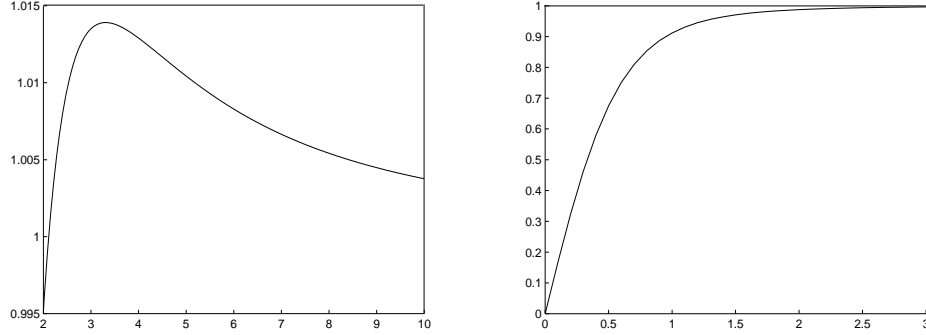


FIGURE 2. Graphs of the functions $\frac{4}{\pi}H_{\mathbb{S}^3}(\mu)$ and $\frac{1}{\pi^2}F_{\mathbb{T}^3}(\mu)$.

the main task being specifying μ_0 . Since $f(x)$ is monotone decreasing, $\int_{a_1}^{a_2} f(x)dx > a_2 f(a_2)$, and the left-hand side is greater than

$$\frac{1}{\mu^2} \frac{1}{(1 + \frac{2}{\mu^2})^k} > \frac{1}{\mu^2} \left(1 - \frac{2k}{\mu^2}\right) = t - 2kt^2 =: L_k(t), \quad t = \mu^{-2}. \quad (5.2)$$

For the right-hand side of (5.1) with $f''(x) = k(k+1)/(x+1)^{k+2}$ and ξ in (4.15) satisfying $\xi > n(n-1)/\mu^2 > ((n-1)/\mu)^2$ we have

$$-\sum_{n=2}^{\infty} R_{a_n, a_{n+1}}(f) = \frac{2k(k+1)}{3\mu^2} \frac{1}{\mu} \sum_{n=2}^{\infty} \frac{(n/\mu)^3}{(\xi_n + 1)^{k+2}}, \quad (5.3)$$

and

$$\begin{aligned} & \frac{1}{\mu} \sum_{n=2}^{\infty} \frac{(n/\mu)^3}{(\xi_n + 1)^{k+2}} < \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{((n+1)/\mu)^3}{((n/\mu)^2 + 1)^{k+2}} = \\ & = \frac{1}{\mu} \sum_{n=1}^{\infty} g_1(n/\mu) + \frac{3}{\mu^2} \sum_{n=1}^{\infty} g_2(n/\mu) + \frac{3}{\mu^3} \sum_{n=1}^{\infty} g_3(n/\mu) + \frac{1}{\mu^4} \sum_{n=1}^{\infty} g_4(n/\mu), \end{aligned}$$

where $g_j(x) = \frac{x^{4-j}}{(x^2+1)^{k+2}}$, $j = 1, 2, 3, 4$. The function $g_1(x)$ has a unique global maximum attained at $x_0 = \left(\frac{3}{2k+1}\right)^{1/2}$. Therefore

$$\begin{aligned} & \frac{1}{\mu} \sum_{n=1}^{\infty} g_1(n/\mu) < x_0 g_1(x_0) + \int_{x_0}^{\infty} g_1(x) dx = \\ & = \frac{9(2k+1)^k}{(2k+4)^{k+2}} + \frac{1}{2k(k+1)} \frac{(5k+4)(2k+1)^k}{(2k+4)^{k+1}} =: G_1(k). \end{aligned}$$

Similarly (replacing x_0 in the integral by 0)

$$\begin{aligned}\frac{1}{\mu} \sum_{n=1}^{\infty} g_2(n/\mu) &< \frac{(k+1)^{k+1/2}}{(k+2)^{k+2}} + \frac{1}{2} \frac{\Gamma(3/2)\Gamma(k+1/2)}{\Gamma(k+2)} =: G_2(k), \\ \frac{1}{\mu} \sum_{n=1}^{\infty} g_3(n/\mu) &< \frac{(2k+3)^{k+3/2}}{(2k+4)^{k+2}} + \frac{1}{2} \frac{1}{k+1} =: G_3(k), \\ \frac{1}{\mu} \sum_{n=1}^{\infty} g_4(n/\mu) &< \frac{1}{2} \frac{\Gamma(1/2)\Gamma(k+3/2)}{\Gamma(k+2)} =: G_4(k).\end{aligned}$$

which gives that the right-hand side in (5.1) is less than

$$\frac{2k(k+1)}{3} (G_1(k)t + 3G_2(k)t^{3/2} + 3G_3(k)t^2 + G_4(k)t^{5/2}) =: R_k(t)$$

and $R_{3/2}(t) = 0.5317 \cdot t + 1.5844 \cdot t^{3/2} + 3.2851 \cdot t^2 + 1.3333 \cdot t^{5/2}$. Obviously, $L_{3/2}(t) = t - 3t^2 \geq R_{3/2}(t)$ for $t \in [0, t_0]$, where t_0 is the first root of the equation $L_{3/2}(t) - R_{3/2}(t) = 0$. We find that $t_0 = 0.0387$. Accordingly, (5.1) holds for all $\mu \geq \mu_0 = (1/t_0)^{1/2} = 5.0833$. Explicitly calculating the integral on the left-hand side of (5.1) and estimating the series involving g_2 and g_3 in the same way as g_1 we have $R_{3/2}(t) = 0.5317 \cdot t + 0.90074 \cdot t^{3/2} + 2.8054 \cdot t^2 + 1.3333 \cdot t^{5/2}$ and therefore can improve the estimate: $\mu_0 = 3.9229$. \square

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